

Synthetic foundations of cevian geometry, II: The center of the cevian conic

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1 Introduction.

In this paper we continue the investigation begun in Part I [mm1], and study the conic $\mathcal{C}_P = ABCPQ$ on the five points A, B, C, P, Q , where $Q = K \circ \iota(P)$ is the isotomcomplement of P , defined to be the complement of the isotomic conjugate of P with respect to ABC . Here, as in Part I, P is any point not on the extended sides of ABC or its anticomplementary triangle. This conic is defined whenever P does not lie on a median of triangle ABC . We show first that the symmetrically defined points $P' = \iota(P)$ and $Q' = K \circ \iota(P') = K(P)$ also lie on this conic, as well as 6 other points that can be given explicitly (see Theorem 2.1 and Figure 1).

Recall from Part I that the map T_P is the unique affine map which takes triangle ABC to the cevian triangle DEF of P with respect to ABC . The map $T_{P'}$ is defined in the same way for the point P' . In Theorem 2.4 we show that if P and P' are ordinary points and do not lie on a median of ABC , then

$$\eta T_P = T_{P'} \eta, \tag{1}$$

where η is the harmonic homology (affine reflection, see [cox3]) whose center is the infinite point V_∞ on the line PP' and whose line of fixed points is the line GV , where G is the centroid of ABC and $V = PQ \cdot P'Q'$. Thus, T_P and $T_{P'}$ are conjugate maps in the affine group. We prove this formula

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synthetically by proving an interesting relationship between the centroids G_1 and G_2 of the cevian triangles DEF and $D_3E_3F_3$ of the points P and P' , respectively. Lemma 2.5 shows that G is the midpoint of the segment G_1G_2 and $\eta(G_1) = G_2$.

After we introduce the affine map

$$\lambda = T_{P'} \circ T_P^{-1}$$

in Section 3, where T_P and $T_{P'}$ are the affine maps considered in Part I, we show that the 6 points on \mathcal{C}_P mentioned above are the images of the vertices A, B, C under λ and λ^{-1} (Theorem 3.4). The mapping λ leaves the conic \mathcal{C}_P invariant as a set (Theorem 3.2), and is the main affine map considered in this paper. The relation (1) allows us to write the map λ as

$$\lambda = \eta \circ (T_P \circ \eta \circ T_P^{-1}) = \eta_1 \circ \eta_2,$$

where both maps η_1 and η_2 are harmonic homologies. Using this representation we prove in Theorem 4.1 that the center $Z = Z_P$ of the conic \mathcal{C}_P is the intersection

$$Z = GV \cdot T_P(GV),$$

and that when \mathcal{C}_P is a parabola or an ellipse, Z is the unique fixed point of λ in the extended plane. When \mathcal{C}_P is a hyperbola, Z is the unique ordinary fixed point of λ . In the latter case, λ also fixes the two points at infinity on the asymptotes of \mathcal{C}_P .

At the end of the paper we interpret the mapping λ as an isometry on the model of hyperbolic geometry whose points are the interior points of the conic \mathcal{C}_P and whose lines are Euclidean chords.

In Part III [mm3] of this series of papers we will prove that the point Z is a generalized Feuerbach point. In particular, this will show that Z is the Feuerbach point of triangle ABC when P is the Gergonne point and P' is the Nagel point of ABC (see [ac] and [ki]). Our Theorem 4.1 therefore gives a representation of the Feuerbach point as the intersection of two lines. Corollary 4.2 shows that these two lines are GV and $T_P(GV) = G_1J$, where J is the midpoint of PQ . A third line through Z is the line G_2J' , where $G_2 = T_{P'}(G)$ and J' is the midpoint of $P'Q'$. (See Figure 5 in Section 4.) In this case \mathcal{C}_P is a hyperbola, and the Feuerbach point Z is the unique ordinary fixed point of $\lambda = T_{P'} \circ T_P^{-1}$.

We mention one more fact that we prove along the way. The mapping

$$\mathcal{S}' = T_{P'}T_PT_{P'}^{-1}T_P^{-1}$$

is always a translation, which implies (Corollary 2.8 and Figure 4) that the triangles $A_3B_3C_3 = T_P(D_3E_3F_3)$ and $A'_3B'_3C'_3 = T_{P'}(DEF)$ are always congruent triangles.

We refer to Part I [mm1] for the notation that we use throughout this series of papers; to [ac], [wo], [y1], or [y2] for definitions in triangle geometry; and to [cox1] and [cox2] for classical results from projective geometry.

2 The conic $ABCPQ$ and the affine mapping T_P .

We start by proving

Theorem 2.1. *If P does not lie on the sides of triangle ABC or its anticomplementary triangle, and not on a median of ABC , then there is a conic \mathcal{C}_P on the points A, B, C, P, Q, P', Q' . This conic also passes through the points*

$$A_0P \cdot D_0Q', \quad B_0P \cdot E_0Q', \quad C_0P \cdot F_0Q', \quad (2)$$

and

$$A'_0P' \cdot D_0Q, \quad B'_0P' \cdot E_0Q, \quad C'_0P' \cdot F_0Q.$$

Proof. The condition that P is not on a median of ABC ensures that the points P and Q are not collinear with one of the vertices. If P, Q , and A are collinear, for example, then the points D and D_2 coincide, so A_1 and A_2 coincide, meaning that Q' is collinear with A and G (by I, Theorem 3.5). But P is collinear with $K(P) = Q'$ and G , so P would lie on AG . By the same reasoning, P, Q' , and A are not collinear and neither are P, P' and A .

Now the mapping T_P is a projective mapping which takes the pencil of lines x on Q' to the pencil $y = T_P(x)$ on P , since $T_P(Q') = P$. For the lines $x = AQ'$ and $y = DP = AP$ we have $x \cdot y = A$; while $x = BQ'$ and $y = EP = BP$ give $x \cdot y = B$; and $x = CQ'$ and $y = FP = CP$ give $x \cdot y = C$. Thus the pencil x is not perspective to the pencil y , so

Steiner's theorem [cox2, p. 80] implies that the locus of points $x \cdot y$ is the conic $ABCPQ'$. If $x = QQ'$ then $y = QP$, so $x \cdot y = Q$ is also on this conic. Hence the conic $ABCPQ' = ABCQQ'$. Arguing the same with the mapping $T_{P'}$ shows that there is a conic $ABCP'Q = ABCQQ'$. It follows that the conic $\mathcal{C}_P = ABCPQ'$ lies on P' and Q , so that $\mathcal{C}_P = ABCPQ$. This proves the first assertion. Letting $x = D_0Q'$ gives $y = A_0P$, so $A_0P \cdot D_0Q'$ is on \mathcal{C}_P , as are all the other listed intersections. \square

Corollary 2.2. (a) *If Y is any point on the conic \mathcal{C}_P other than P, Q' (respectively Q, P'), then $T_P(Q'Y) = PY$ (resp. $T_{P'}(QY) = P'Y$).*

(b) *The conic \mathcal{C}_P is the locus of points Y for which P, Y , and $T_P(Y)$ are collinear.*

(c) *In particular, $P, P', T_P(P')$, and $T_{P'}(P)$ are collinear (whether P lies on a median or not).*

Proof. Part (a) follows from the definition of the conic \mathcal{C}_P as the locus of intersections $x \cdot y$. For part (b), if Y lies on \mathcal{C}_P , then by part (a), P, Y , and $T_P(Y)$ are collinear. This also clearly holds for $Y = P$ and $Y = Q'$. Conversely, suppose P, Y , and $T_P(Y)$ are collinear. If $Y \notin \{P, Q'\}$, then T_P maps the line $x = Q'Y$ to the line $y = PT_P(Y) = PY$, so $x \cdot y = Y$ lies on \mathcal{C}_P . Part (c) follows from part (b) with $Y = P'$ and from the analogous statement obtained by switching P and P' , as long as P does not lie on a median of ABC . Now assume that $P \neq G$ lies on the median AG . Then P, P', Q , and Q' are all on AG . From the Collinearity Theorem (I, Theorem 3.5) the points D_i for $0 \leq i \leq 4$ are all the same point, so $A_0 = A_2 = A_3$, the midpoint $EF \cdot AG$ of EF ; and similarly, the points A'_i equal $A'_0 = A'_3$, the midpoint $E_3F_3 \cdot AG$ of E_3F_3 . From P' on AD_3 it follows that $T_P(P')$ is on $DA_3 = D_0A_2 = AG$. Similarly, $T_{P'}(P)$ lies on $AG = PP'$. \square

Remarks. 1. We know from I, Theorems 3.13 and 2.4 that the lines D_0Q' , E_0Q' , F_0Q' in (1) are the cevians for the point Q' with respect to the anticevian triangle $A'B'C'$ of Q , since $A'Q', B'Q'$, and $C'Q'$ pass through the midpoints of the sides of ABC . See Theorem 3.4 below.

2. Part (b) of the corollary allows for an easy computation of an equation for the conic \mathcal{C}_P .

We will assume the hypothesis of Theorem 2.1 anytime we make use of the conic $\mathcal{C}_P = ABCPQ$. Alternatively, we could define \mathcal{C}_P to be a degenerate

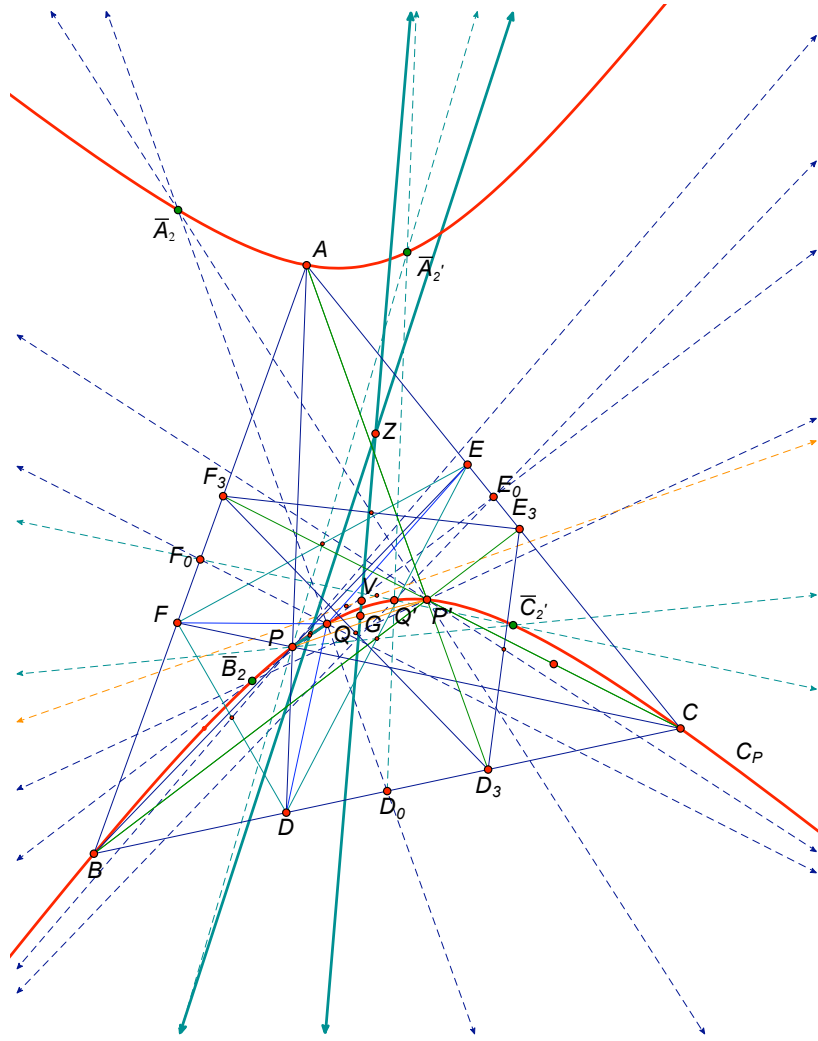


Figure 1: Cevian conic, with points $\bar{A}_2 = A'_0P' \cdot D_0Q$, $\bar{A}'_2 = A_0P \cdot D_0Q'$, etc.

conic, the union of a median and a side of ABC , when P does lie on a median. In that case, P, Q, P', Q' are all on the median.

Assume now that the point P is ordinary and does not lie on a median of triangle ABC or on $\iota(l_\infty)$. Then the points P' and Q are also ordinary. We shall use the conic \mathcal{C}_P to prove an interesting relationship between the maps T_P and $T_{P'}$. First note that the lines PP' and QQ' are parallel, since $K(PGP') = Q'GQ$. The midpoint of QQ' is clearly the complement of the midpoint of PP' , so the line joining them passes through the centroid G . Since the quadrangle $PP'QQ'$ is inscribed in \mathcal{C}_P , its diagonal triangle is a self-polar triangle for \mathcal{C}_P [cox2, p. 75]. The vertices of this self-polar triangle are $PQ' \cdot P'Q = G$, $PQ \cdot P'Q' = V$, and $PP' \cdot QQ' = V_\infty$, a point on the line at infinity. Hence, the polar of G is the line VV_∞ , which is the line through V parallel to PP' . The polar of V_∞ is GV . Since the segment QQ' is parallel to PP' and half its length, Q is the midpoint of segment PV , so V is the reflection of P in Q , and the reflection of P' in Q' . Considering the quadrangle $VQGG'$, it is not hard to see that GV passes through the midpoints of segments PP' and QQ' , since these midpoints are harmonic conjugates of V_∞ with respect to the point pairs (QQ') and (PP') .

Let $Z = Z_P$ be the center of the conic $\mathcal{C}_P = ABCPQ$, so Z is the pole of the line at infinity. Since V_∞ lies on the polar of Z , Z must lie on GV . Since PP' and QQ' are parallel chords on the conic \mathcal{C}_P , the line through their midpoints passes through Z [cox1, p. 111]. Hence we have:

Proposition 2.3. *Assume that the ordinary point P does not lie on a median of triangle ABC or on $\iota(l_\infty)$.*

- (a) *The points $G, V = PQ \cdot P'Q'$, and $V_\infty = PP' \cdot QQ'$ form a self-polar triangle with respect to the conic $\mathcal{C}_P = ABCPQ$.*
- (b) *The center Z of the conic \mathcal{C}_P lies on the line $GV = G(PQ \cdot P'Q')$, which is the polar of the point V_∞ .*
- (c) *The line $GV = GZ$ passes through the midpoints of the parallel chords PP' and QQ' .*
- (d) *The harmonic homology μ_G with center G and axis VV_∞ maps the conic \mathcal{C}_P to itself. In other words, if a line GY intersects the conic in points X_1 and X_2 and $GY \cdot VV_\infty = X_3$, then the cross-ratio $(X_1X_2, GX_3) = -1$.*

(e) The line VV_∞ is the same as the line joining the anti-complements of P and P' . Thus, the polar of G is the line $K^{-1}(PP')$.

(f) The point V is the midpoint of the segment joining $K^{-1}(P)$ and $K^{-1}(P')$.

Proof. We have already proven parts (a)-(c). Part (d) is immediate from the fact that a conic is mapped into itself by any homology whose center is the pole of its axis [cox2, p. 76, ex. 4]. For part (e), note that $\mu_G(P) = Q'$, since PQ' is a chord of the conic containing G . Hence $(PQ', GX_3) = -1$, where $GP \cdot VV_\infty = X_3$; this implies $PX_3 = -2X_3Q'$, which means that Q' is the midpoint of PX_3 . But $Q' = K(P)$, so $K(X_3) = P$. Thus X_3 is the anti-complement of P . Similarly, the anti-complement of P' lies on VV_∞ , and this proves part (e). Part (f) follows from the fact that the midpoint M of PP' lies on the line GV , so that

$$K^{-1}(M) = K^{-1}(PP') \cdot K^{-1}(GV) = K^{-1}(PP') \cdot GV = V,$$

since the line GV is an invariant line of the complement map. \square

Let $\eta = \eta_P$ be the harmonic homology whose center is V_∞ and whose axis is its polar GV . The map η is an affine reflection [cox3, p. 203], since it fixes the line GV and maps a point Y to the point Y' with the property that YY' is parallel to VV_∞ (or PP') and $YY' \cdot GV$ is the midpoint of YY' . The map η is an involution on the extended plane. It takes the conic \mathcal{C}_P to itself and interchanges the point pairs (PP') and (QQ') , since the line GV passes through the midpoints of the chords PP' and QQ' and both lines PP' and QQ' lie on V_∞ . Hence this homology induces an involution of points on \mathcal{C}_P .

Remark. It is not hard to show that the map η commutes with the complement map: $K\eta = \eta K$.

We shall now prove

Theorem 2.4. *Assume that the ordinary point P does not lie on a median of triangle ABC or on $\iota(l_\infty)$. Then the maps T_P and $T_{P'}$ satisfy the equation $\eta T_P = T_{P'}\eta$, and so are conjugate to each other in the affine group.*

To prove this theorem we need a lemma, which is of interest in its own right.

Lemma 2.5. *Let $G_1 = T_P(G)$ and $G_2 = T_{P'}(G)$ be the centroids of the cevian triangles DEF and $D_3E_3F_3$ of P and P' . Then the centroid G of ABC is the midpoint of the segment G_1G_2 , which is parallel to PP' . In other words, $\eta(G_1) = G_2$ (when P and P' are ordinary).*

Proof. We first show that the line G_1G_2 lies on G and is parallel to PP' . To begin with, assume P and P' are ordinary. Using I, Corollary 3.3, we know that the points $Q, G_1, T_P(P')$ are collinear, with G_1 one-third of the way from Q to $T_P(P')$. Since G is collinear with Q and P' and one-third of the way along QP' , the triangles G_1QG and $T_P(P')QP'$ are similar (SAS). Hence the line G_1G is parallel to the line $T_P(P')P' = PP'$ (Corollary 2.2). Switching the roles of P and P' gives that G_2G is also parallel to PP' , which implies that $G_1G = G_2G$, proving the claim.

If $P' = Q$ is infinite, then P and Q' are ordinary, so we still get $G_2G \parallel PP'$. Applying the map T_P and using I, Theorem 3.14 gives that $T_P(G_2G) \parallel T_P(PP')$, i.e., $GG_1 \parallel PP'$, because $T_P(G_2) = T_P T_{P'}(G) = K^{-1}(G) = G$ and $T_P(PP') = T_P(PQ) = T_P(P)Q \parallel PQ = PP'$. Hence we get $G_1G = G_2G$, as before. A similar argument works if $P = Q'$ is infinite.

We now show that G is the midpoint of G_1G_2 . (Cf. Figure 3.) Consider the sequence of triangles DEF , D_0EF , D_3EF with centroids G_1, G_{01}, G_{11} lying on the respective lines A_0D, A_0D_0, A_0D_3 (since A_0 is the midpoint of EF). By the properties of the centroid we know that the segment G_1G_{11} is the image of the segment DD_3 under a dilatation with center A_0 and ratio $1/3$. Since D_0 is the midpoint of DD_3 it follows that the vector $\overrightarrow{G_1G_{01}}$ is $\frac{1}{2}$ the vector $\overrightarrow{G_1G_{11}}$.

Next consider passing from triangle D_3EF with centroid G_{11} to the triangle D_3E_3F with centroid G_{12} . Considering the dilatation with ratio $1/3$ from the midpoint of D_3F shows that the vector $\overrightarrow{G_{11}G_{12}}$ is $\frac{1}{3}$ the vector $\overrightarrow{EE_3}$. In the same way, if we move from triangle D_0EF with centroid G_{01} to triangle D_0E_0F with centroid G_{02} , the vector $\overrightarrow{G_{01}G_{02}}$ is $\frac{1}{3}$ the vector $\overrightarrow{EE_0}$, and therefore $\frac{1}{2}$ the vector $\overrightarrow{G_{11}G_{12}}$.

Finally, in passing from triangle D_3E_3F with centroid G_{12} to triangle $D_3E_3F_3$ with centroid G_2 , the vector $\overrightarrow{G_{12}G_2}$ is $\frac{1}{3}$ the vector $\overrightarrow{FF_3}$. Similarly, in passing from D_0E_0F with centroid G_{02} to triangle $D_0E_0F_0$ with centroid

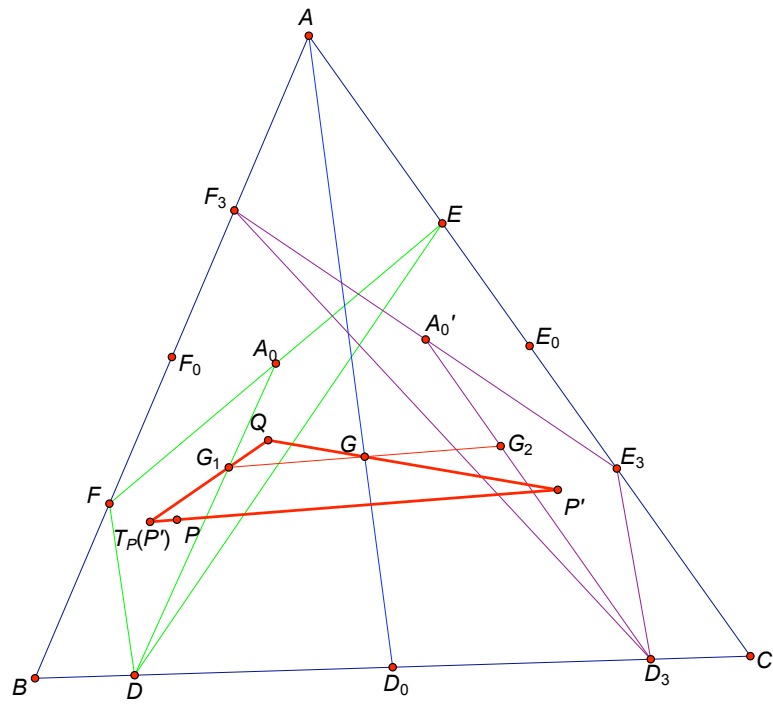


Figure 2: $G_1G_2||PP'$

G , the vector $\overrightarrow{G_0G}$ is $\frac{1}{3}$ of $\overrightarrow{FF_0}$. Hence, the vector $\overrightarrow{G_0G}$ is $\frac{1}{2}$ the vector $\overrightarrow{G_{12}G_2}$.

It follows that in passing from triangle DEF to $D_0E_0F_0$, the centroid experiences a displacement represented by the vector

$$\overrightarrow{G_1G_{01}} + \overrightarrow{G_{01}G_{02}} + \overrightarrow{G_{02}G} = \frac{1}{2}(\overrightarrow{G_1G_{11}} + \overrightarrow{G_{11}G_{12}} + \overrightarrow{G_{12}G_2}),$$

which is $\frac{1}{2}$ the vector displacement from G_1 to G_2 . Hence, $G_1G = \frac{1}{2}G_1G_2$, which proves that G is the midpoint of G_1G_2 . \square

Proof of Theorem 2.4. We check that $\eta T_P(Y) = T_{P'}\eta(Y)$ for three non-collinear ordinary points Y . This holds for $Y = G$ by the lemma. It also holds for the points Q and Q' , since I, Theorems 3.2 and 3.7 imply that

$$\eta T_P(Q) = \eta(Q) = Q' = T_{P'}(Q') = T_{P'}\eta(Q)$$

and

$$\eta T_P(Q') = \eta(P) = P' = T_{P'}(Q) = T_{P'}\eta(Q').$$

The points Q, Q' , and G are clearly not collinear, since G is the intersection of the diagonals in the trapezoid $PP'Q'Q$ (and no three of these points lie on a line because they all lie on the conic \mathcal{C}_P). This implies the theorem, since ηT_P and $T_{P'}\eta$ are affine maps. \square

Remark. The map η is the unique involution ψ in the affine group satisfying $\psi T_P = T_{P'}\psi$.

For the corollary, recall that the point $X = AA_3 \cdot BB_3$ is the fixed point of the map $\mathcal{S}_1 = T_P T_{P'}$, and $X' = AA'_3 \cdot BB'_3$ is the fixed point of $\mathcal{S}_2 = T_{P'} T_P$.

Corollary 2.6. *Assume that the ordinary point P does not lie on a median of triangle ABC or on $\iota(l_\infty)$. If X is an ordinary point, then $\eta(X) = X'$ and XX' is parallel to PP' . Thus the line joining the P -ceva conjugate of Q and the P' -ceva conjugate of Q' is parallel to PP' .*

Proof. We have $T_{P'} T_P(\eta(X)) = T_{P'}\eta(T_P(X)) = \eta T_P T_{P'}(X) = \eta(X)$, which shows that $\eta(X)$ is an ordinary fixed point of $\mathcal{S}_2 = T_{P'} T_P$. Hence, $\eta(X) = X'$. This implies the assertion, by I, Theorems 3.8 and 3.10. \square

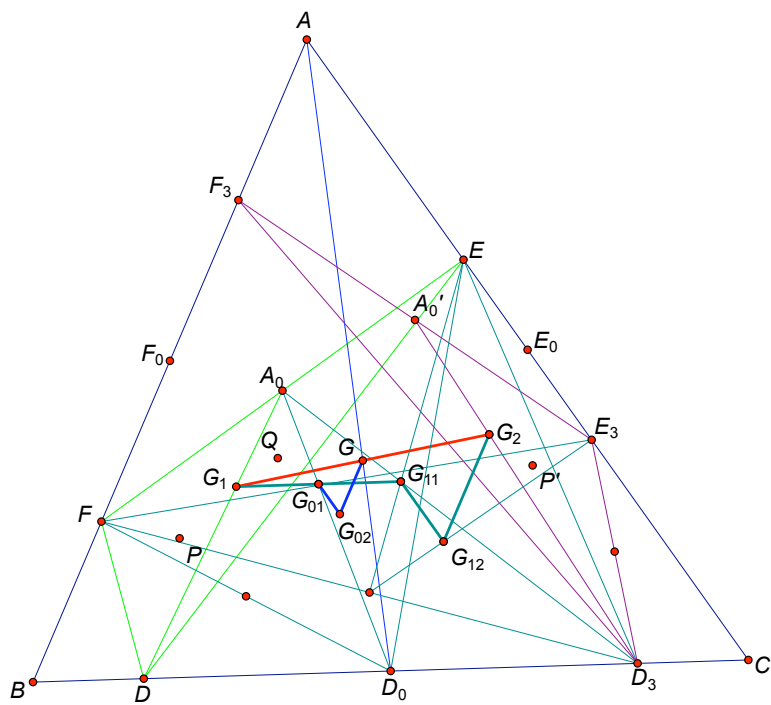


Figure 3: G is the midpoint of G_1G_2

Theorem 2.7. *The commutator $\mathcal{S}' = T_{P'}T_PT_{P'}^{-1}T_P^{-1}$ is always a translation, in the direction PP' by the distance $T_P(P')P' \cong PT_{P'}(P)$, if P and P' are ordinary; and by the distance $3|G_1G|$, if P or P' is infinite.*

Proof. For notational convenience write T_1 for T_P and T_2 for $T_{P'}$. We first note that

$$\mathcal{S}'(T_1(P')) = T_2T_1T_2^{-1}(P') = T_2T_1(Q) = T_2(Q) = P'$$

and similarly

$$\mathcal{S}'(P) = T_2T_1T_2^{-1}(Q') = T_2T_1(Q') = T_2(P).$$

From this computation and Corollary 2.2, the mapping \mathcal{S}' fixes the line PP' .

By I, Theorem 3.8 and Corollary 3.11(c) we may assume that both $\mathcal{S}_1 = T_1T_2$ and $\mathcal{S}_2 = T_2T_1$ are homotheties, since otherwise the assertion is trivial. If P and P' are ordinary points, then $\mathcal{S}_2(Q) = P'$ implies that X', P' , and Q are collinear, so part (b) of the same corollary implies that

$$\frac{X'P'}{X'Q} = \frac{T_1(X'P')}{T_1(X'Q)} = \frac{XT_1(P')}{XQ} = \frac{XP}{XQ};$$

the last equality being a consequence of $\mathcal{S}_1(Q) = T_1(P')$ and $\mathcal{S}_1(Q') = P$. This equation shows that the similarity ratios of \mathcal{S}_1 and \mathcal{S}_2 are equal and the mapping $\mathcal{S}' = \mathcal{S}_2\mathcal{S}_1^{-1}$ is an isometry which fixes l_∞ pointwise. It follows that \mathcal{S}' is either a half-turn or a translation.

First assume that P does not lie on a median of ABC . Then $\eta(T_1(P')) = T_2(\eta(P')) = T_2(P)$, so that $T_1(P')$ and $T_2(P)$ are both inside or both outside the segment PP' , and $T_1(P')P' \cong PT_2(P)$ since η preserves lengths along PP' . If the midpoints of segments $T_1(P')P'$ and $PT_2(P)$ are M_1 and M_2 , then $\eta(M_1) = M_2$. If $M_1 = M_2$ then M_1 is the midpoint of PP' , impossible since $T_1(P') \neq P = T_1(Q')$. Thus, $M_1 \neq M_2$, and \mathcal{S}' fixes the line PP' but is not a half-turn. Hence, \mathcal{S}' is a translation. This proves the assertion in the case that P is not on a median.

Now assume that $P \neq G$ lies on the median AG . Then P, P', Q , and Q' are distinct points on AG . (This follows easily from the fact that P, Q are on the opposite side of G on line AG from P', Q' and $T_P(Q) = Q$ and $T_P(Q') = P$. See I, Theorems 3.2 and 3.7.) From I, Theorem 3.5 the points A_i for $0 \leq i \leq 4$ are all the same point $A_0 = A_3$, the midpoint $EF \cdot AG$ of

EF ; and similarly, the points A'_i equal $A'_0 = A'_3$, the midpoint $E_3F_3 \cdot AG$ of E_3F_3 . Also, $T_1(P')$ and $T_2(P)$ are on $PP' = AG$, so T_1, T_2 and \mathcal{S}' all fix the line $AG = AD = PP'$.

If P lies in the interior of triangle ABC , then it is easy to see that $E = T_1(B)$ and $E_3 = T_2(B)$ are on the same side of the line AD as point C , and hence that T_1 and T_2 interchange the sides of this line. Therefore, $B_3 = T_1(E_3)$ and $B'_3 = T_2(E)$ are on the same side of line $AD = AG$, which implies that segments $B_3B'_3$ and $A_3A'_3$ do not intersect. Since

$$\mathcal{S}'(A_3B_3C_3) = T_2T_1T_2^{-1}(D_3E_3F_3) = T_2T_1(ABC) = A'_3B'_3C'_3,$$

we see that \mathcal{S}' is not a half-turn, and is therefore a translation. On the other hand, if P is exterior to ABC , points E and E_3 are on opposite sides of the line AD , so one of T_1 and T_2 interchanges the sides of AD and the other leaves both sides of AD invariant. Hence $B_3 = T_1(E_3)$ and $B'_3 = T_2(E)$ are on the same side of the line AD and we get the same conclusion as before.

Finally, assume that $P' = Q$ is infinite. Using I, Theorem 3.14, we have as in the proof of Lemma 2.5 that $T_1(G) = G_1, T_1(G_2) = T_1T_2(G) = K^{-1}(G) = G$ and $T_1(Q) = Q$. Since G is the midpoint of the segment G_1G_2 , the fundamental theorem of projective geometry implies that T_1 acts as translation along the line GG_1 . In this situation the map $\mathcal{S}_1 = K^{-1}$ fixes G , so $X = G$, while $\mathcal{S}_2 = T_2T_1$ fixes $G_2 = X'$. (See I, Theorem 3.10 and Corollary 3.11.) Furthermore, $\mathcal{S}_2(G) = T_2T_1(G) = T_1^{-1}K^{-1}(G_1) = T_1^{-1}(G_3)$, where $G_3 = K^{-1}(G_1) = T_1^{-1}(G_2)$. Hence, $\mathcal{S}_2(G) = T_1^{-2}(G_2)$, and the similarity ratio of the homothety \mathcal{S}_2 is -2 , which is equal to the similarity ratio of $\mathcal{S}_1 = K^{-1}$. Once again, $\mathcal{S}' = \mathcal{S}_2\mathcal{S}_1^{-1} = \mathcal{S}_2K$ is an isometry. It is now straightforward to verify that the map \mathcal{S}' , which is the commutator of T_1^{-1} and K^{-1} , is equal to the translation T_1^{-3} on the line GG_1 . For example, $\mathcal{S}'(G) = \mathcal{S}_2K(G) = T_1^{-2}(G_2) = T_1^{-3}(G)$, while $\mathcal{S}'(G_3) = \mathcal{S}_2(G_1) = T_1^{-1}K^{-1}T_1(G_1) = T_1^{-4}(G_2) = T_1^{-3}(G_3)$. Hence, the points on the line GG_1 experience a translation and not a half-turn. If instead, $P = Q'$ is infinite, we apply the same argument to the inverse of \mathcal{S}' . This completes the proof. \square

Corollary 2.8. (a) *The triangles $A_3B_3C_3$ and $A'_3B'_3C'_3$ are always congruent to each other. (See Figure 4.)*

(b) *If P and P' are ordinary, the length $|T_P(P')P|$ of the segment $T_P(P')P$ is equal to $|QQ'| |XP| / |XQ'|$.*

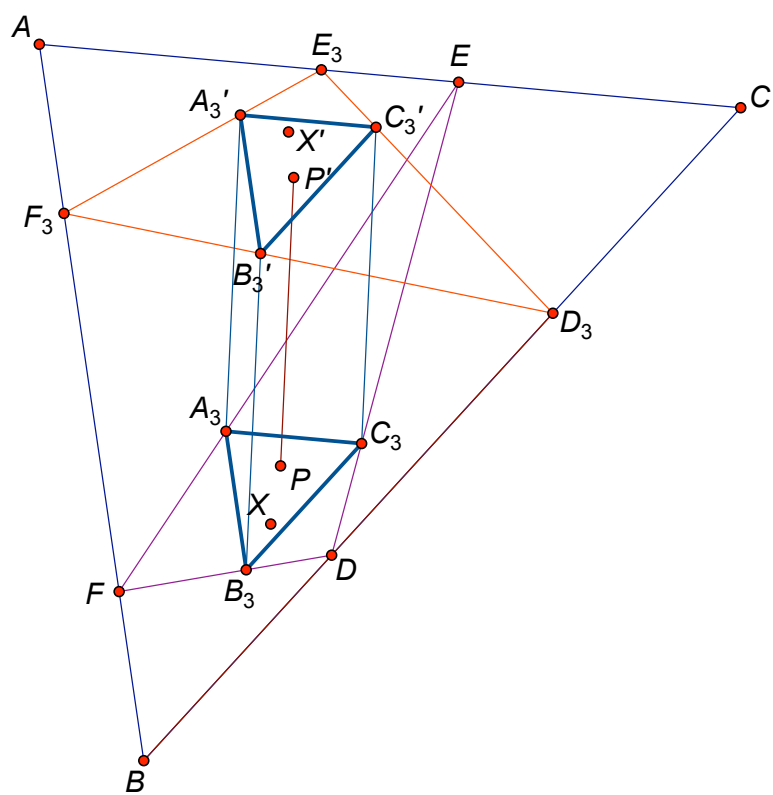


Figure 4: $A_3B_3C_3 \cong A'_3B'_3C'_3$

Proof. Part (a) follows from $\mathcal{S}'(A_3B_3C_3) = A'_3B'_3C'_3$. Part (b) follows from $\mathcal{S}_1(Q) = T_1T_2(Q) = T_1(P')$ and $\mathcal{S}_1(Q') = P$, since \mathcal{S}_1 is a homothety with fixed point X . \square

3 The affine map λ .

We now set $\lambda = T_{P'}T_P^{-1}$, the unique affine map taking the cevian triangle DEF for P to the cevian triangle $D_3E_3F_3$ for P' . We will show that λ maps the conic \mathcal{C}_P to itself!

We use the fact that the diagonal triangle of any quadrangle inscribed in \mathcal{C}_P is self-polar [cox1, p. 73]. Since A, B, C are on \mathcal{C}_P , if Y is any other point on \mathcal{C}_P , the cevian triangle whose vertices are the traces of Y on the sides of ABC is a self-polar triangle. We apply this to the point $Y = P$, obtaining that DEF is a self-polar triangle.

Associated to the vertex D of the self-polar triangle DEF is the harmonic homology η_D whose center is D and whose axis is EF . We know P and A are collinear with D and A_4 (on EF) by the Collinearity Theorem. Furthermore, η_D maps the conic to itself. Since P and A are on the conic \mathcal{C}_P , the definition of η_D implies that $\eta_D(A) = P$ and hence that $(AP, DA_4) = -1$. This proves

Proposition 3.1. *The point sets $AA_4PD_1, BB_4PE_1, CC_4PF_1$ are harmonic sets.*

This is the key fact we use to prove the following.

Theorem 3.2. *If P does not lie on a median of triangle ABC , then the map $\lambda = T_{P'}T_P^{-1}$ takes the conic \mathcal{C}_P to itself: $\lambda(\mathcal{C}_P) = \mathcal{C}_P$.*

Proof. Consider the anticevian triangle of Q , which, by I, Corollary 3.11, is $A'B'C' = T_{P'}^{-1}(ABC)$. The cevian triangle for Q with respect to this triangle is ABC , so Q plays the role of P in the above discussion and ABC plays the role of DEF . We have that $AQ \cdot BC = D_2$, so Proposition 3.1 implies that $A'D_2QA$ is a harmonic set. Now we map this set by T_P . We have that $T_P(A'D_2QA) = T_P(A')A_2QD$, so $T_P(A')$ is the harmonic conjugate of Q with respect to D and A_2 . But A_2 is on EF , so the above discussion implies that $\eta_D(Q) = T_P(A')$. Since Q is on \mathcal{C}_P , so is $T_P(A') = T_P T_{P'}^{-1}(A) = \lambda^{-1}(A)$.

Similar arguments apply to the other vertices, so λ^{-1} maps A, B, C to points on \mathcal{C}_P . Now it is easy to see that $\lambda^{-1}(P') = Q$ and $\lambda^{-1}(Q') = P$, using I, Theorems 3.2 and 3.7. Hence λ^{-1} maps 5 points on \mathcal{C}_P to 5 other points on \mathcal{C}_P , so we must have $\lambda^{-1}(\mathcal{C}_P) = \mathcal{C}_P$. This implies the assertion. \square

If P is ordinary and not on $\iota(l_\infty)$, then the map $\lambda = T_{P'}T_P^{-1} = \eta T_P \eta T_P^{-1}$ is the product of η and $\eta\lambda = T_P \eta T_P^{-1}$, both of which are involutions (harmonic homologies) on the extended plane, and both of which fix the conic \mathcal{C}_P . The involution $T_P \eta T_P^{-1}$ interchanges P and Q . Its axis of fixed points is the line $T_P(GV)$ and its center $T_P(V_\infty) = T_P(PP' \cdot QQ')$ lies on $T_P(QQ') = PQ$, so that the midpoint of segment PQ lies on $T_P(GV)$.

The map $T_P \eta T_P^{-1}$ is the map corresponding to η for the conic $T_P(\mathcal{C}_P) = T_P(ABCQ'Q) = DEFPQ$, which also lies on the points $T_P(P)$ and $T_P(P')$. Since Q is the complement of $T_P(P')$ with respect to the triangle DEF (I, Corollary 3.3), the conic $DEFPQ$ equals \mathcal{C}_R for triangle DEF and the point $R = T_P(P)$. This is because the cevian triangle for $T_P(P)$ in DEF is $A_1B_1C_1$, the cevian triangle for $T_P(P')$ in DEF is $A_3B_3C_3$, and the points $T_P(P)$ and $T_P(P')$ are isotomic conjugates with respect to DEF .

Remark. It is easy to show that $\eta\lambda\eta = \lambda^{-1}$. Moreover, the map λ is never a projective homology. (Hint: show $T_P(V_\infty)$ is never on the line GV and use [cox2, p. 56, ex. 4].)

The last theorem has several interesting consequences.

Theorem 3.3. *If P does not lie on a median of triangle ABC , then the 6 vertices of the anticevian triangles for Q and Q' (with respect to ABC) lie on the conic $T_P^{-1}(\mathcal{C}_P) = T_{P'}^{-1}(\mathcal{C}_P)$, along with the points Q and Q' . This conic is $\mathcal{C}_Q = A'B'C'QQ'$ for the anticevian triangle of Q , which is the same as $\mathcal{C}_{Q'}$ for the anticevian triangle of Q' . Moreover, the vertices of the anticevian triangle of R with respect to ABC , for any point R on $T_P^{-1}(\mathcal{C}_P)$, lie on $T_P^{-1}(\mathcal{C}_P)$.*

Proof. In the proof of the previous theorem we showed that $T_P(A') = \lambda^{-1}(A)$ lies on the conic \mathcal{C}_P , so A' lies on the conic $T_P^{-1}(\mathcal{C}_P)$, as do B' and C' , the other vertices of the anticevian triangle for Q . Similarly, the vertices of the anticevian triangle for Q' lie on the conic $T_{P'}^{-1}(\mathcal{C}_P)$. But these two conics are the same conic, since $T_{P'}T_P^{-1}(\mathcal{C}_P) = \mathcal{C}_P$ implies $T_P^{-1}(\mathcal{C}_P) = T_{P'}^{-1}(\mathcal{C}_P)$. That Q and Q' also lie on this conic follows from $T_P^{-1}(Q) = Q$ and $T_P^{-1}(P) = Q'$.

Theorem 3.12 of Part I implies the second assertion. Since the triangle DEF is self-polar for \mathcal{C}_P , it follows that ABC is a self-polar triangle for $T_P^{-1}(\mathcal{C}_P)$. The last assertion of the theorem follows from the general projective fact, that for any point R on a conic \mathcal{C} , the vertices of the anticevian triangle of R , with respect to any self-polar triangle for the conic, also lie on \mathcal{C} . (See [cox2, p. 91, ex. 3].) \square

Note that the map η fixes the conic $T_P^{-1}(\mathcal{C}_P)$, since $\eta T_P^{-1}(\mathcal{C}_P) = T_{P'}^{-1}\eta(\mathcal{C}_P) = T_{P'}^{-1}(\mathcal{C}_P)$.

Remark. If P is the orthocenter of ABC , then Q' is the circumcenter, Q is the symmedian point [gr1, Thm. 7], and P' is the point $X(69)$, the symmedian point of the anticomplementary triangle (see [ki]). In this case, the anticevian triangle of Q is the tangential triangle [wo], so Theorem 3.3 implies that $T_P^{-1}(\mathcal{C}_P)$ is the Stammler hyperbola [y3, p. 21].

Theorem 3.4. *Assume that the ordinary point P does not lie on a median of triangle ABC or on $\iota(l_\infty)$.*

(a) *The last-named points of Theorem 2.1 are*

$$A_0P \cdot D_0Q' = \lambda^{-1}(A), \quad B_0P \cdot E_0Q' = \lambda^{-1}(B), \quad C_0P \cdot F_0Q' = \lambda^{-1}(C);$$

$$A'_0P' \cdot D_0Q = \lambda(A), \quad B'_0P' \cdot E_0Q = \lambda(B), \quad C'_0P' \cdot F_0Q = \lambda(C).$$

(b) *Moreover, the lines A_0P, D_0Q', DQ, A'_3P' are concurrent at the point $\lambda^{-1}(A)$, and $A'_0P', D_0Q, D_3Q', A_3P$ are concurrent at the point $\lambda(A)$, with similar statements holding for the other points in (a).*

Proof. From the proof of Theorem 3.2 we have that $\lambda^{-1}(A)$ is on the intersection of the line DQ with the conic \mathcal{C}_P and is distinct from Q . The proof of Theorem 2.1 together with $T_{P'}(DQ) = T_{P'}(D'_3Q) = A'_3P'$ shows that $DQ \cdot A'_3P'$ is also on \mathcal{C}_P . Now $DQ \cdot A'_3P'$ is not the point Q ; otherwise Q would lie on A'_3P' , which would imply the point X' is on A'_3P' , since X' is collinear with Q and P' (I, Theorem 3.8). However, the point A is on $X'A'_3$ by the definition of X' (I, Theorem 3.5), so Q and P' would be collinear with A . This contradicts the assumption that P is not on AG . Hence, $DQ \cdot A'_3P' = \lambda^{-1}(A)$, since DQ intersects the conic in exactly two points.

On the other hand, I, Corollary 3.11 and Theorems 3.12 and 2.4 give that the point $A' = T_{P'}^{-1}(A)$ is on both lines AQ and D_0Q' , so we have that

$$\lambda^{-1}(A) = T_P T_{P'}^{-1}(A) = T_P(AQ \cdot D_0Q') = DQ \cdot A_0P.$$

Note that $DQ \cdot A_0P \neq P$ since Q does not lie on $PD = AP$ (see the proof of Theorem 2.1). Finally, $A_0P \cdot D_0Q'$ is on \mathcal{C}_P , by Theorem 2.1, and this intersection is not P , since P and Q' are collinear with G and G is not on D_0Q' . Hence, $A_0P \cdot D_0Q' = \lambda^{-1}(A)$. Therefore, the lines A_0P, D_0Q', DQ , and A'_3P' meet at the point $\lambda^{-1}(A)$. This proves the first assertion in (b). The second assertion follows immediately upon reversing the roles of P and P' . This gives two of the equalities in part (a), and the others follow by the same reasoning applied to the other vertices. \square

Corollary 3.5. *Under the assumptions of the theorem, the two quadrangles $PQQ'P'$ and $A_0DD_0A'_3$ are perspective, as are quadrangles $PQQ'P'$ and $A_3D_0D_3A'_0$. (See Part I, Figure 6, where $Y = \lambda(A)$.)*

Theorem 3.6. *The translation $\mathcal{S}' = T_{P'}T_PT_P^{-1}T_{P'}^{-1}$ of Theorem 2.7 maps the conic $DEFPQ$ to the conic $D_3E_3F_3P'Q'$. In other words, these two conics are congruent.*

Proof. From Theorem 3.3 we have, again with $T_1 = T_P$ and $T_2 = T_{P'}$, that

$$\begin{aligned} \mathcal{S}'(DEFPQ) &= T_2T_1T_2^{-1}(ABCQ'Q) = T_2T_1(T_2^{-1}(\mathcal{C}_P)) \\ &= T_2T_1(T_1^{-1}(\mathcal{C}_P)) = T_2(\mathcal{C}_P) = D_3E_3F_3P'Q'. \end{aligned}$$

\square

4 The center Z of \mathcal{C}_P .

In this section we study the center $Z = Z_P$ of the conic \mathcal{C}_P , which we will recognize as a generalized Feuerbach point in Part III.

Theorem 4.1. *Assume that the ordinary point P does not lie on a median of ABC or on $\iota(l_\infty)$. Then the center Z of the conic \mathcal{C}_P is given by $Z = GV \cdot T_P(GV)$. If \mathcal{C}_P is a parabola or an ellipse, Z is the unique fixed point in*

the extended plane of the affine mapping $\lambda = T_P T_P^{-1}$. If \mathcal{C}_P is a hyperbola, the infinite points on the asymptotes are also fixed, and these are the only other invariant points of λ .

Proof. Since the map λ leaves invariant the conic and the line at infinity, it fixes the pole of this line, which is Z .

To prove uniqueness write the map $\lambda = \eta_1 \eta_2$ as the product of the harmonic homologies $\eta_1 = \eta$ and $\eta_2 = \eta \lambda = T_P \eta T_P^{-1}$ (see the discussion following Theorem 3.2). The center of η_1 is V_∞ , lying on the line PP' , and the center of η_2 is $T_P(V_\infty)$, lying on the line $T_P(QQ') = QP$. If R is any ordinary fixed point of λ , then $\eta_1(R) = \eta_2(R) = R'$. If R is distinct from R' , this implies that RR' is parallel to both lines PP' and QP , which is impossible. If $R = R'$, then R is fixed by both η_1 and η_2 , so R must be the intersection $GV \cdot T_P(GV)$ of the axes of the two maps. This proves that $Z = GV \cdot T_P(GV)$ if Z is ordinary.

Suppose now that the point $Z (= GV \cdot l_\infty)$ is an infinite point (so \mathcal{C}_P is a parabola) and R is another infinite fixed point. Then $\eta_1(R) = \eta_2(R) = R'$ is also an infinite fixed point of λ , since $\lambda \eta(R) = \eta \lambda^{-1}(R) = \eta(R)$. If $R \neq Z, R'$, then λ fixes three points on l_∞ and is therefore the identity on l_∞ . But $\lambda(PQ) = Q'P'$, and $PQ \cdot P'Q'$ is the ordinary point V (Proposition 2.3f), implying that PQ cannot be parallel to $Q'P'$ and the line at infinity cannot be a range of fixed points. Thus, since Z on GV is fixed by η_1 we have $R = R'$, implying that R is fixed by η_1 and η_2 . We know that $V_\infty \neq T_P(V_\infty)$ (because $T_P(QQ') = QP$). Since R is fixed by η_1 but different from Z , then $R = V_\infty$, which must lie on the axis $T_P(GV)$ in order to be fixed by η_2 . Now $T_P(GV)$ lies on the point $G_1 = T_P(G)$ and on the midpoint J of segment PQ (see the comments following Theorem 3.2). The point G_1 lies on the line GV_∞ through G parallel to PP' (Lemma 2.5), but J is not on GV_∞ , since this line divides the segment PQ in the ratio 2:1. Hence, $T_P(GV) = G_1J$ cannot possibly lie on the point V_∞ . Thus $R = Z$ and Z is the only infinite invariant point of λ .

We have already shown that the only possible ordinary fixed point of λ is the point $Y = GV \cdot T_P(GV)$. We now show this point is infinite when Z is infinite. Assume Y is an ordinary point. Then Y is on the line GV with Z , so $GV = YZ$ is an invariant line; hence its pole V_∞ is also invariant. Now use the fact that $\lambda(P) = Q'$ and $\lambda(Q) = P'$. Since V_∞ is invariant the line PP' must be mapped to the line QQ' . Since the conic is also invariant, P' must map to the second point of intersection of QQ' with the conic, which is

Q . Hence, λ interchanges P' and Q , implying that the line $P'Q$ is invariant; hence the infinite point on this line is invariant. This point is clearly not V_∞ ; it is also not Z because $P'Q \cdot GV = G$, so that if Z were on $P'Q$ then $P'Q = GV$, which is not the case since P' and Q are not fixed by the map η . This shows that λ has three invariant points on the line at infinity, which contradicts the fact that λ is not the identity on l_∞ . Therefore, λ has no ordinary fixed point in this case, and we conclude that $Z = GV \cdot T_P(GV)$ if Z is infinite. Hence, $Z = GV \cdot T_P(GV)$ in all cases.

Suppose next that Z is ordinary and \mathcal{C}_P is a hyperbola. Let R_1 and R_2 be the infinite points on the two asymptotes. Since the point V_∞ is the pole of the axis GV of the map η_1 , the point $I_1 = GV \cdot l_\infty$ is conjugate to V_∞ on l_∞ and $GV = ZI_1$ and ZV_∞ are conjugate diameters. By a theorem in Coxeter [cox1, p. 111, 8.82], these two conjugate diameters are harmonic conjugates with respect to the asymptotes. (Note that GV is never an asymptote: since GVV_∞ is the diagonal triangle of the quadrangle $PQQ'P'$, the pole V_∞ of GV never lies on GV .) Hence we have the harmonic relation $H(R_1R_2, I_1V_\infty)$. By the definition of η_1 this implies that $\eta_1(R_1) = R_2$. In the same way, $T_P(V_\infty)$ is the center and $T_P(GV)$ is the axis of the harmonic homology η_2 . Since η_2 fixes the conic, it fixes the pole of the line $T_P(GV)$ with respect to \mathcal{C}_P ; this pole is the center $T_P(V_\infty)$ since $T_P(V_\infty)$ is the only fixed point of η_2 off the line $T_P(GV)$. Hence we have the harmonic relation $H(R_1R_2, I_2T_P(V_\infty))$, where $I_2 = T_P(GV) \cdot l_\infty$, and this implies that $\eta_2(R_1) = R_2$. Therefore, $\lambda(R_1) = \eta_1\eta_2(R_1) = \eta_1(R_2) = R_1$ and $\lambda(R_2) = R_2$. There cannot be any other fixed points since λ induces a non-trivial map on the line l_∞ .

The only case left to consider is the case when \mathcal{C}_P is an ellipse. In this case Z is an interior point of the conic and any line through Z intersects the conic in two points. Suppose that the infinite point on the line ZR_1 is fixed by λ , where R_1 lies on the conic. As before we have $\eta_1(ZR_1) = \eta_2(ZR_1)$, so $R_2 = \eta_1(R_1)$ lies on the conic, along with $R_3 = \eta_2(R_1)$, and Z, R_2 , and R_3 are collinear. The points R_2 and R_3 are distinct because Z is the only ordinary fixed point of λ . Now line R_1R_2 is on the point V_∞ and R_1R_3 is on the point $T_P(V_\infty)$, so $R_1R_2R_3$ is a triangle. Furthermore, $T_P(GV)$ is on the midpoint of segment R_1R_3 . Since Z is the center, it is the midpoint of the segment R_2R_3 , and therefore the line $T_P(GV)$, which lies on Z and the midpoint of R_1R_3 , is parallel to R_1R_2 . This implies that $T_P(GV)$ lies on V_∞ , which we showed above is never the case. Therefore, Z is the only fixed point of λ in this case. This completes the proof of the theorem. \square

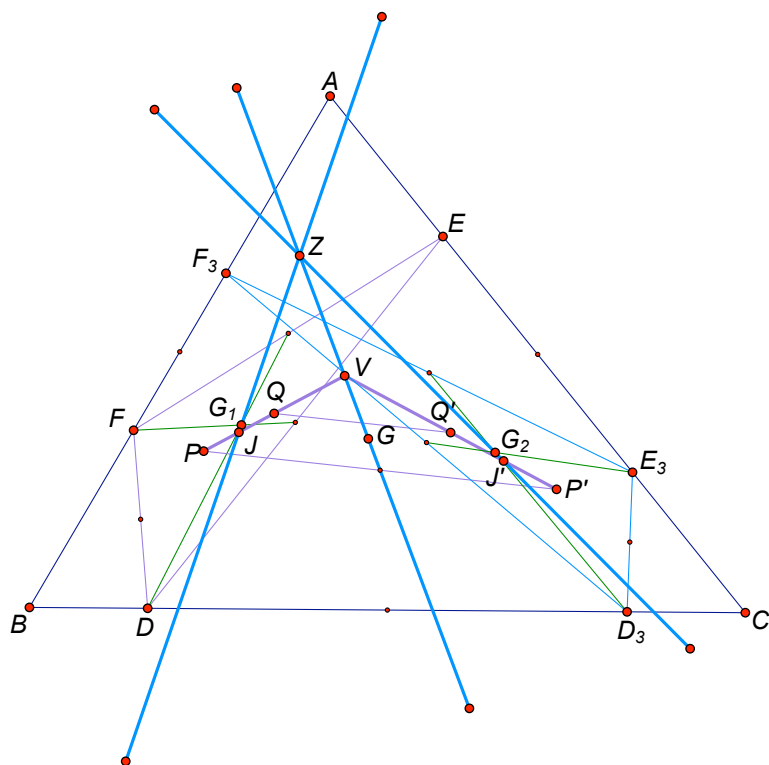


Figure 5: The center Z of the conic \mathcal{C}_P

Corollary 4.2. (a) *The point $Z = GV \cdot G_1J$, where G_1 is the centroid of the cevian triangle of P and J is the midpoint of segment PQ .*
(b) *The lines $GV, T_P(GV) = G_1J$, and $T_{P'}(GV) = G_2J' = \eta(G_1J)$ are concurrent at Z . (See Figure 5.)*

Theorem 4.3. *Assume that the ordinary point P does not lie on a median of ABC but does lie on the Steiner circumellipse $\iota(l_\infty)$. Then the point Z is one-third of the distance from the point G to the point $G_1 = T_P(G)$ on the line GG_1 and is the only ordinary fixed point of the mapping λ . In this case \mathcal{C}_P is a hyperbola and the line $l = GG_1$ is an asymptote.*

Proof. This will follow from the discussion in the last paragraph of the proof of Theorem 2.7, according to which T_P acts as translation on $l = GG_1$ and $\lambda = T_{P'}T_P^{-1} = T_P^{-1}K^{-1}T_P^{-1}$. First of all, it follows easily from this representation of λ that λ fixes the point Z_1 on l for which $GZ_1 : GG_1 = 1 : 3$, and that this Z_1 is the only fixed point of λ on this line.

Furthermore, the mapping T_P interchanges the sides of the line GG_1 . This is because the point D_2 is on the line $AQ = AP' = AD_2$, which is parallel to the fixed line l ; and $A_2 = T_P(D_2)$ is on the opposite side of l , because A_2 is a vertex of the anticomplementary triangle (I, Corollary 3.14) and the point G lies on segment AA_2 . Since K interchanges the sides of the line l , it follows that the mapping λ has the same property. Thus, λ has no ordinary fixed points off of l , and Z_1 is its only ordinary fixed point.

On the other hand, λ fixes the center Z of the conic \mathcal{C}_P . If $Z \neq Z_1$, then Z is infinite and \mathcal{C}_P is a parabola. Now the infinite point Q lies on l and \mathcal{C}_P , so $Z = Q$, and the line l must intersect the parabola in a second point, which would have to be a fixed point of λ , since l and \mathcal{C}_P are both invariant under λ . This second point on $l \cap \mathcal{C}_P$ is therefore the point Z_1 . Now all the points (other than Z_1) of the conic \mathcal{C}_P lie on one side of the tangent line l_1 of Z_1 . Note that $l \neq l_1$; otherwise Q on l_1 would imply l_∞ is on Z_1 by the polarity. If U and V are points on \mathcal{C}_P on either side of Z_1 , then they lie on opposite sides of l and the intersection $UV \cdot l$ is mapped to another point on l by λ , which must lie on the same side of l_1 . This would say that λ , which fixes the tangent l_1 , leaves invariant both sides of this line. However,

$$\lambda(G) = T_P^{-1}K^{-1}T_P^{-1}(G) = T_P^{-1}K^{-1}(G_2) = T_P^{-1}T_P^2(G) = G_1,$$

yet G and G_1 are on opposite sides of l_1 . This contradiction shows that $Z = Z_1$, proving the first assertion. This implies that \mathcal{C}_P is a hyperbola and

the line $l = GG_1$ is an asymptote, since l is on Z and cannot intersect \mathcal{C}_P at a point other than Q for the same reason as before - such an intersection would have to be a second ordinary fixed point of λ . This completes the proof. \square

To view Theorem 4.1 from a different perspective, consider the model of hyperbolic geometry whose points are the interior points of the conic \mathcal{C}_P and whose lines are the intersections of Euclidean chords on the conic with the interior of \mathcal{C}_P . Consider the case when \mathcal{C}_P is an ellipse. The involutions $\eta_1 = \eta$ and $\eta_2 = T_P \eta T_P^{-1}$ fix the conic \mathcal{C}_P and map the interior of \mathcal{C}_P to itself. Furthermore, the points R and $\eta(R) = R'$ lie on a line “perpendicular” to GV in this model, since RR' lies on the pole of GV . (See Greenberg [gre, p. 309].) Note that GV contains points that are interior to \mathcal{C}_P , because Z is an interior point in this case. Since η preserves cross-ratios it is a hyperbolic isometry. Thus, η represents reflection across the diameter GZ in this model [gre, pp. 341-343]. The mapping $\eta_2 = \eta\lambda = T_P \eta T_P^{-1}$ is also a hyperbolic isometry fixing at least two points and thus a reflection across the diameter $T_P(GV) = G_1J = G_1Z$ [gre, p. 412]. Thus, the map $\lambda = \eta_1\eta_2$ represents a hyperbolic rotation about the point Z .

When \mathcal{C}_P is a hyperbola, then Z is an exterior point. If the lines GV and $T_P(GV)$ are secant lines, they both pass through the pole of the line at infinity (also a secant line) and are therefore perpendicular to l_∞ in the hyperbolic model. In this case λ represents a translation along l_∞ . If the axis GV does not intersect the conic, then the homology η_1 interchanges the sides of the line l_∞ (in the model). Since the asymptotes separate the lines GV and ZV_∞ (see the proof of Theorem 4.1), the point V_∞ is a fixed point of η_1 in the hyperbolic plane, and all lines through V_∞ are invariant. Hence, η_1 is a half-turn about V_∞ . If the axis $T_P(GV)$ is a secant line, then λ is an indirect isometry with no fixed points and an invariant line, and is therefore a glide [gre, p. 430]. If neither axis is a secant line, λ is the product of two half-turns and is therefore a translation. When \mathcal{C}_P is a parabola, Z is a point on the conic and therefore an ideal point in the hyperbolic model. In this case every line through Z intersects the conic, so GV and $T_P(GV)$ are secants and λ is a parallel displacement [gre, p. 424].

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